Calculus 140, section 5.4 The Fundamental Theorem of Calculus
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Beginning with an explanation, and following with a proof (which you should read through for a full understanding), the text presents the following Theorem relating an integral to an antiderivative.

Theorem 5.12: "Let f be continuous on an interval I (containing more than one point) and let c be any point in I. Define G by the equation $G(x) = \int_c^x f(t)\, dt$ for all x in I. Then G is differentiable on I, and $G'(x) = f(x)$ for all x in I. In particular, f has an antiderivative on I."

Side note: The use of the "dummy variable" t in the integral was necessary, since x is the variable for the function G.

Example A: Let $G(x) = \int_1^x t e^t\, dt$ for all x. Find $G'(x)$.

Theorem 5.13 [Fundamental Theorem of Calculus]: "Let f be continuous on $[a, b]$.

   a. The function G defined by $G(x) = \int_a^x f(t)\, dt$ for x in $[a, b]$ is an antiderivative of f on $[a, b]$.

   b. If F is an antiderivative of f on $[a, b]$, then $\int_a^b f(x)\, dx = F(b) - F(a)$.

Proof:

The importance of the Fundamental Theorem is that it relates differentiation and integration, showing that these two operations are, for all practical purposes, inverses of one another. This is essentially the result published by both Newton and Leibniz.

See Table 5.1 in the text for a few basic functions and their antiderivatives.

Example B: Evaluate $\int_{\pi/6}^{\pi/4} \sec^2 x\, dx$.   *answer*: $1 - \frac{\sqrt{3}}{3}$

Note that the Fundamental Theorem can make use of *any* antiderivative, so we'll generally choose the easiest version, i.e. the one without a constant term "$+\, C$". This works because, as in the proof above, we'd always get "$+\, C - C = 0$" anyway.

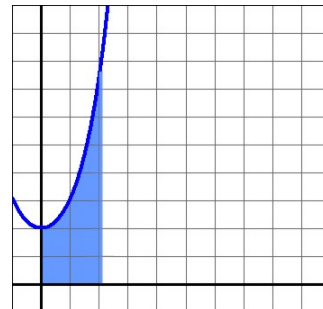5.2 Example B revisited: Evaluate $\int_{2}^{5} 2x\, dx$. *answer*: 21
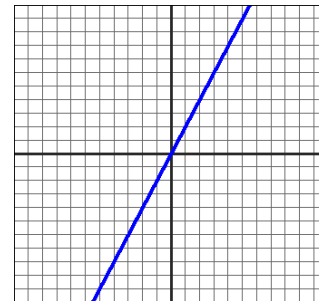
5.1 Example B revisited: Find the area under the curve $y = 2\sqrt{x}$ on the interval [2, 7].

answer: $\frac{4}{3}\sqrt{7^3} - \frac{4}{3}\sqrt{2^3}$
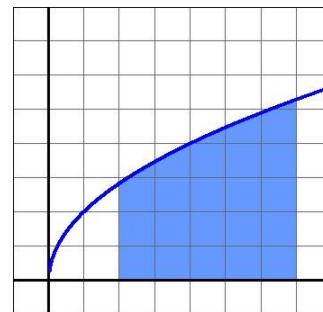
Example C: Find the area under the curve $y = e^x + e^{-x}$ on the interval [0, ln 8].

answer: $\frac{63}{8}$

Corollary 5.14: "Let f be continuous on [a, b]. Then for any antiderivative F of f, $\int_{b}^{a} f(x)\, dx = F(a) - F(b)$.

The proof is a direct application of Definition 5.6, part 2.

The implication of the Fundamental Theorem and Corollary 5.14 is that $\int_{c}^{d} f(x)\, dx = F(d) - F(c)$ no matter whether we have $c < d$, $c = d$, or $c > d$.

Example D: Evaluate $\int_{\pi/2}^{0} \cos x \, dx$. *answer*: -1

The conclusion of Theorem 5.12 can be stated as $\dfrac{d}{dx} \int_{c}^{x} f(t) \, dt = f(x)$.

Now consider $\int_{g(x)}^{h(x)} f(t) \, dt = F(h(x)) - F(g(x))$ for any antiderivative F of f.

Then, $\dfrac{d}{dx} \int_{g(x)}^{h(x)} f(t) \, dt = \dfrac{d}{dx} [F(h(x)) - F(g(x))]$.

Using the Chain Rule and the fact that $F' = f$, we get $\dfrac{d}{dx} \int_{g(x)}^{h(x)} f(t) \, dt = f(h(x)) * h'(x) - f(g(x)) * g'(x)$.

Example E: Evaluate $\dfrac{d}{dy} \int_{y}^{y^2} e^{t^2} \, dt$. *answer*: $2ye^{y^4} - e^{y^2}$

Now consider a rocket fired upward from the ground at velocity 80 feet per second. Its height as a function of time is given by the function $h(t) = -16t^2 + 80t$. The two times it at height $= 0$ are found by solving
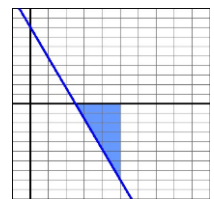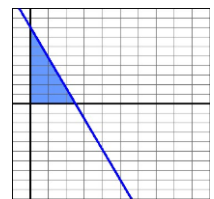$$0 = -16t^2 + 80t = -16t(t - 5) \quad \Rightarrow \quad t = 0 \text{ sec. and } t = 5 \text{ sec.}$$
The velocity of the rocket is given by the function
$$v(t) = h'(t) = -32t + 80.$$

Example F: Find and interpret "area under the curve" for the function $f(t) = -32t + 80$ on the intervals   a) [0, 2.5],   b) [2.5, 5],   and c) [0, 5].
answers: 100; $-$ 100; 0 [*new question answers*: 100; 100; 200]

The important concept here is that while area between the curve and the *x*-axis which is above the *x*-axis is *positive*, area between the curve and the *x*-axis which is below the *x*-axis is *negative*. In an application where negative values make sense, these negative values can be useful. (For example, an object which is falling loses height.)

Another application involves money. For $f(t) = $ <u>rate</u> of income/expenses with respect to time, the antiderivative (area under the curve) would be <u>amount</u> of income/expenses. If $\int_{t_n}^{t_{n+1}} f(t) \, dt > 0$, there was money left over. If

$\int_{t_n}^{t_{n+1}} f(t) \, dt = 0$, the budget balanced exactly. If $\int_{t_n}^{t_{n+1}} f(t) \, dt < 0$, more money was spent than came in.